Name: ....

Teacher: ....

Fort Street High School

2013
Trial HSC Examination
Assessment Task 3

Mathematics Extension 2

Time allowed: 3 hours (plus 5 minutes reading time)

Syllabus	Assessment Area Description and Marking Guidelines	Questions
	Chooses and applies appropriate mathematical techniques in order to solve problems effectively	1-10 13b
E4, E6	Uses efficient techniques for the algebraic manipulation of conic sections and determining features of a wide variety of graphs	11c 12c 13c
E2, E3	Applies appropriate strategies to construct arguments and proofs in the areas of complex numbers and polynomials	11a,b,d 14b
E7, E8	Applies further techniques of integration, such as slicing and cylindrical shells, integration by parts and recurrence formulae, to problems	12a,b 13a 14a,c
E5	Synthesises mathematical solutions to harder problems and communicates them in an appropriate form	15 16

Total Marks 100

Section I 10 marks
Multiple Choice, attempt all questions,
Allow about 15 minutes for this section

Section II 90 Marks
Attempt Questions 11-16,
Allow about 2 hours 45 minutes for this section

General Instructions:
- Questions 11-16 are to be started in a new booklet
- The marks allocated for each question are indicated
- In Questions 11 – 16, show all relevant working.
- Marks may be deducted for careless or badly arranged work.
- Board – approved calculators may be used

Section I	Total 10	Marks
Q1-Q10		
Section II	Total 90	Marks
Q11	/15	
Q12	/15	
Q13	/15	
Q14	/15	
Q15	/15	
Q16	/15	
	Percent	

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Section I

10 marks Attempt Questions 1–10 Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 Let $z = 3 - i$.

What is an expression for $\dfrac{1}{z}$?

(A) $\dfrac{3}{4} + \dfrac{1}{4}i$

(B) $\dfrac{3}{8} + \dfrac{1}{8}i$

(C) $\dfrac{3}{10} + \dfrac{1}{10}i$

(D) $\dfrac{3}{2} + \dfrac{1}{2}i$

2 What are the co-ordinates for the foci of the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$

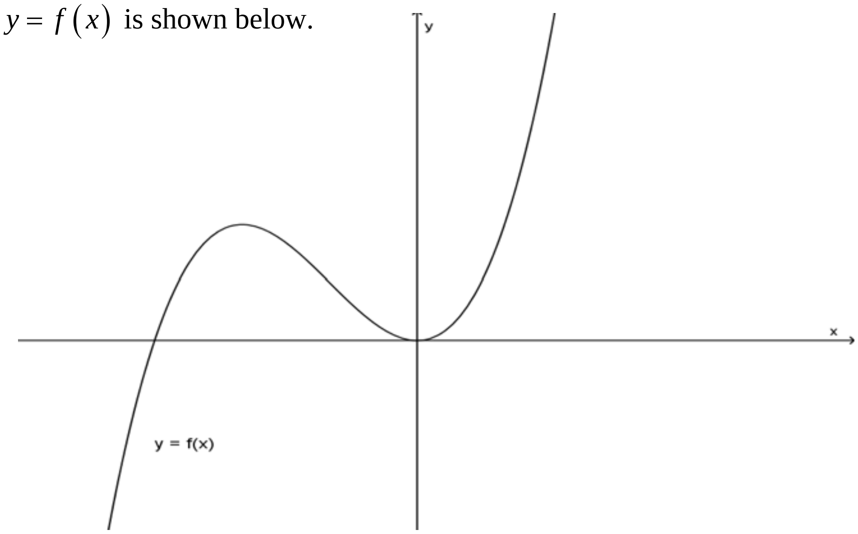
(A) $\left(\pm\sqrt{5}, 0\right)$

(B) $\left(\pm\dfrac{2\sqrt{5}}{3}, 0\right)$

(C) $\left(0, \pm\sqrt{5}\right)$

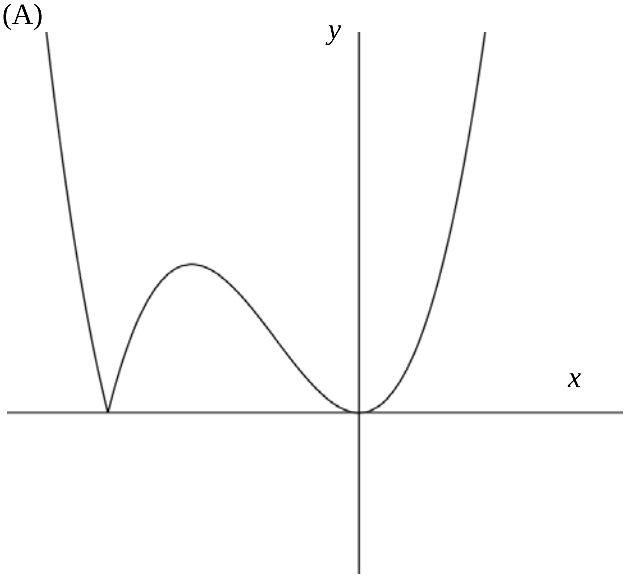
(D) $\left(0, \pm\dfrac{2\sqrt{5}}{3}\right)$

3 The graph of $y = f(x)$ is shown below.

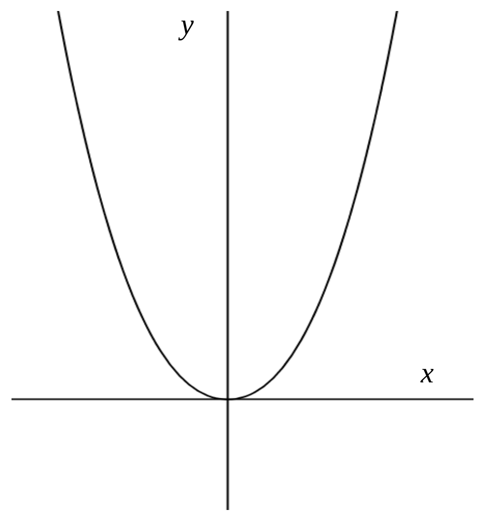


Which of the following graphs best represents $y = f(|x|)$?

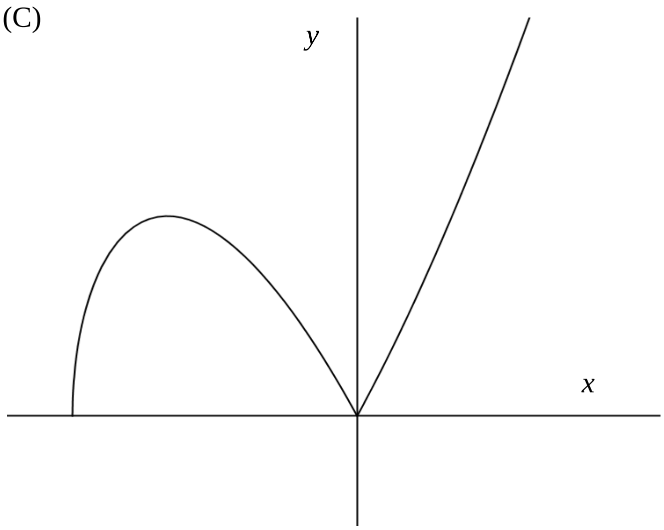
(A)



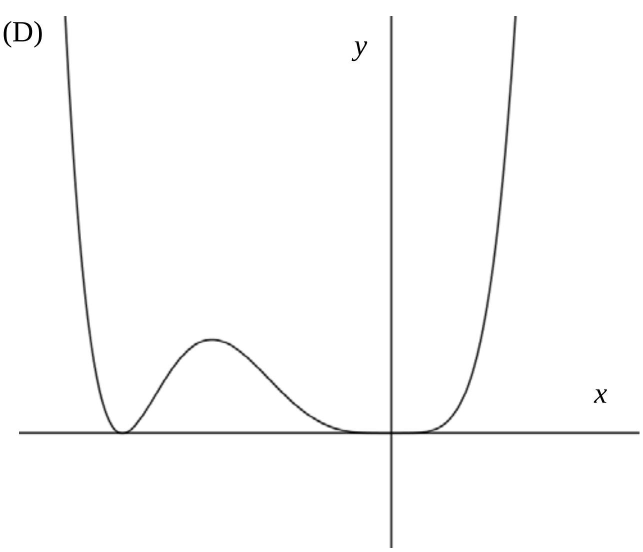
(B)



(C)



(D)



4 Express $z=\sqrt{3}+i$ in modulus-argument form.

(A) $\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}$

(B) $\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$

(C) $2\cos\frac{\pi}{6}+2i\sin\frac{\pi}{6}$

(D) $2\cos\frac{\pi}{3}+2i\sin\frac{\pi}{3}$

5 By using the standard table of integrals evaluate $\int_0^{\frac{\pi}{2}}\frac{\cos x}{\sqrt{4-\sin^2 x}}\,dx$

(A) $\frac{\pi}{2}$

(B) $\frac{\pi}{6}$

(C) $\frac{1}{2}$

(D) $\frac{1}{4}$

6 What is the remainder when x^3+x^2+5x+6 is divided by $x+i$

(A) $7-4i$

(B) $7-6i$

(C) $5-4i$

(D) $5+6i$

7 α, β, γ are the roots of $x^3 - 4x^2 + x - 5 = 0$. An equation that has roots $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ is

(A) $5x^3 + x^2 - 4x - 1 = 0$

(B) $1 - 4x + x^2 - 5x^3 = 0$

(C) $5x^3 + 4x + x^2 - 1 = 0$

(D) $1 + 4x + x^2 - 5x^3 = 0$

8 The point A represents the complex number $-4-3i$. $\angle OZA = 90^\circ$ and $|ZA| = 2|z|$.

Find the complex number represented by the point Z.

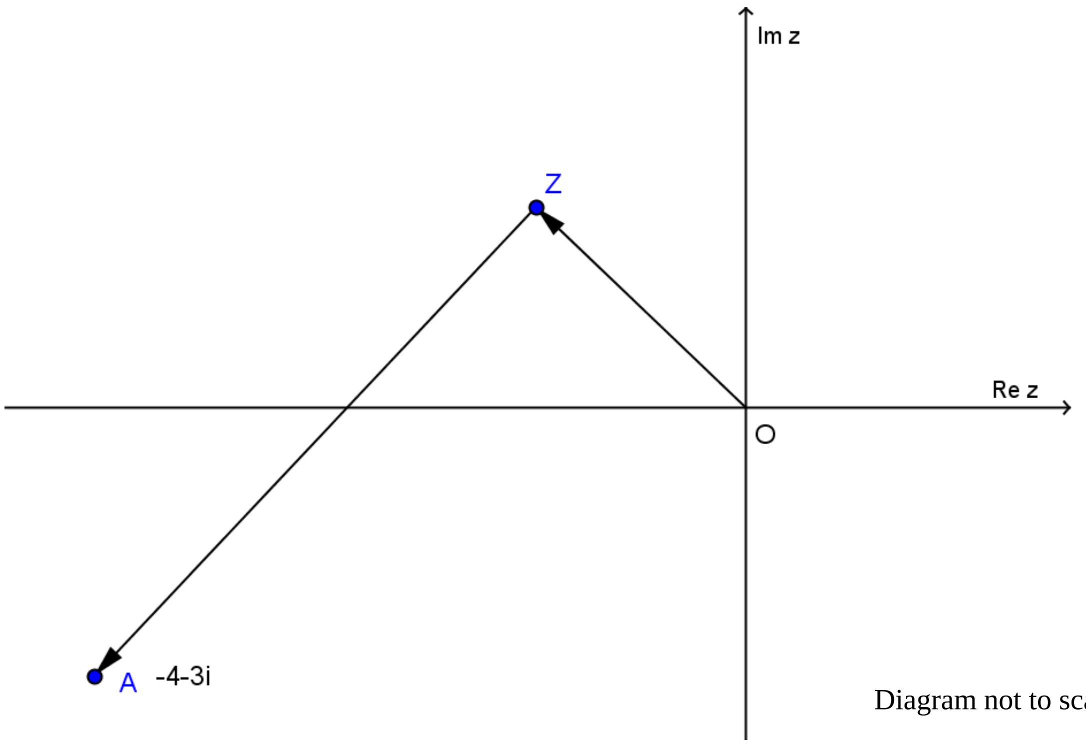


Diagram not to scale.

(A) $-1+\sqrt{2}i$

(B) $-1+2i$

(C) $-2+i$

(D) $-\sqrt{2}+i$

9 A mass of 1 kg is released from rest at the surface of a medium in which the retardation on the mass is proportional to the distance fallen (x). The net force for this motion is $g - kx$ Newtons with the downward direction as positive.

The mass will become stationary after falling how far?

(A) $\dfrac{g}{k}$

(B) $\dfrac{2g}{k}$

(C) $\dfrac{kv}{g}$

(D) $\dfrac{2g}{kv}$

10 What is the approximate value of $\int_0^{\frac{\pi}{2}} \frac{1}{\cos\theta + 2\sin\theta + 3}\, d\theta$?

(A) $0 \cdot 322$

(B) $0 \cdot 785$

(C) $1{\cdot}150$

(D) $1{\cdot}571$

Section II

90 marks Attempt Questions 11–16 Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11. (15 marks) Use a *separate* writing booklet.

(a) On an Argand diagram, sketch the locus of the points z such that $|z+1| = |z-i|$. **2**

(b) Find the square roots of $5+2\sqrt{6}i$. **2**

(c) A curve is defined by the implicit equation $3x^2 + y^2 - 2xy - 8x + 2 = 0$.

(i) Show that $\dfrac{dy}{dx} = \dfrac{3x-y-4}{x-y}$. **2**

(ii) Sketch the curve showing any turning points. **3**

(d) (i) Show that $z^n + z^{-n} = 2\cos n\theta$. **2**

(ii) Hence solve $5z^4 - z^3 + 6z^2 - z + 5 = 0$. **4**

Question 12. (15 marks) Use a *separate* writing booklet.

(a) Find $\int \frac{\sec^4\theta}{\tan\theta}\, d\theta$. **3**

(b) Let $I_n = \int \left(x^2 + a^2\right)^n \, dx$, where n is an integer.

(i) Show that $I_n = \frac{x\left(x^2+a^2\right)^n}{2n+1} + \frac{2na^2}{2n+1} I_{n-1}$. **3**

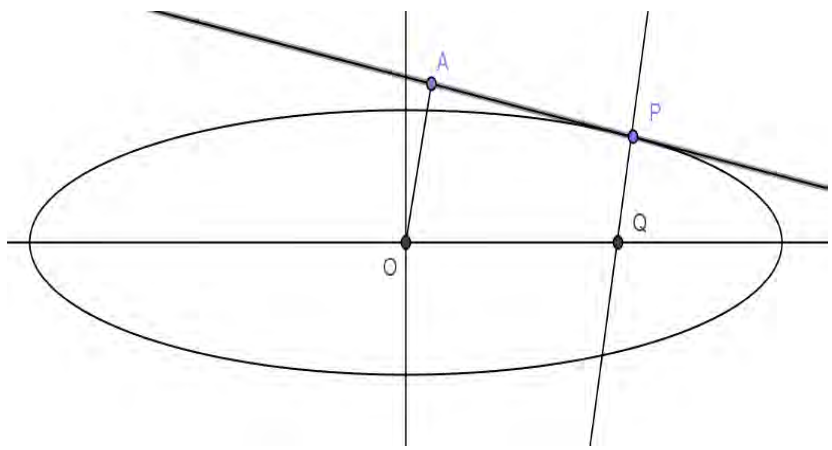
(ii) Hence evaluate $\int_0^1 \left(x^2+2\right)^3 \, dx$. **2**

(c) An ellipse has the equation $x^2 + 16y^2 = 25$.

(i) Find the gradient of the ellipse at point $P(3,1)$. **2**

(ii) Find the equation of the tangent and normal to the ellipse at *P*. **2**

(iii) The normal to the ellipse, at point *P*, meets the major axis at *Q*. A line from the centre, *O* to the tangent at *P* meets at right angles at point *A*. **3**



Show that the value of $PQ \times OA$ is equal to the square of the semi-minor axis.

Question 13. (15 marks) Use a *separate* writing booklet.

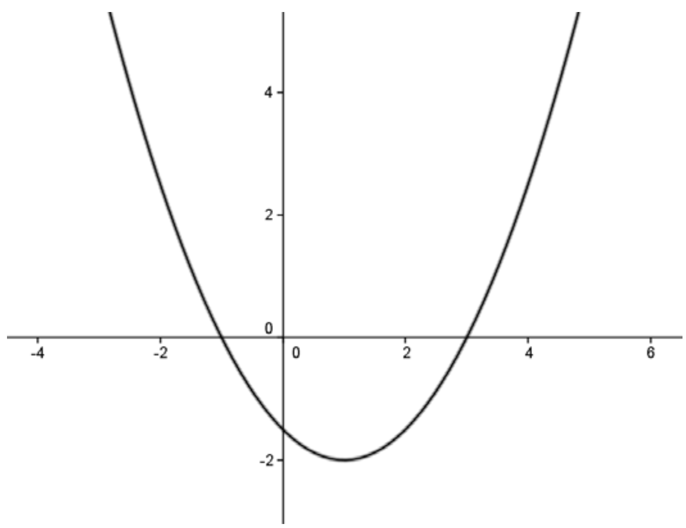
(a) The area bounded by the curve $y = x(2-x)$ and the x-axis is rotated about the y-axis.

(i) By using the method of cylindrical shells show that the volume, $V = 2\pi\int_0^2 xy \; dx$ **2**

(ii) Hence find the volume of the solid. **3**

(b) High tide for a harbour occurs at 5 a.m. and low tide at 11:20 a.m. with the depths of the harbour at high and low tide 30 and 10 metres respectively. If the tidal motion is assumed to be in simple harmonic motion find the latest time before noon that a ship needing 25 metres clearance could exit the harbour. **4**

(c) The diagram shows the graph of $f(x) = \frac{1}{2}(x-3)(x+1)$.



Use the above graph to draw a one-third page sketch, showing any intercepts or asymptotes, of

(i) $y = \dfrac{1}{f(x)}$ (Show the coordinates of the turning point) **3**

(ii) $y = e^{-x} f(x)$ (Do **NOT** show the coordinates of the turning point) **3**

Question 14. (15 marks) Use a *separate* writing booklet.

(a) (i) Find real numbers *a*, *b* and *c* such that $\dfrac{1}{x^3-1}=\dfrac{a}{x-1}+\dfrac{bx+c}{x^2+x+1}$. **2**

(ii) Hence find $\int \dfrac{dx}{x^3-1}$. **3**

(b) α,β,γ are the roots of the equation $x^3-px^2+qx-r=0$.

(i) Find an expression for the following in terms of *p*,*q*,*r*. **3**

(1) $\alpha+b+\gamma$

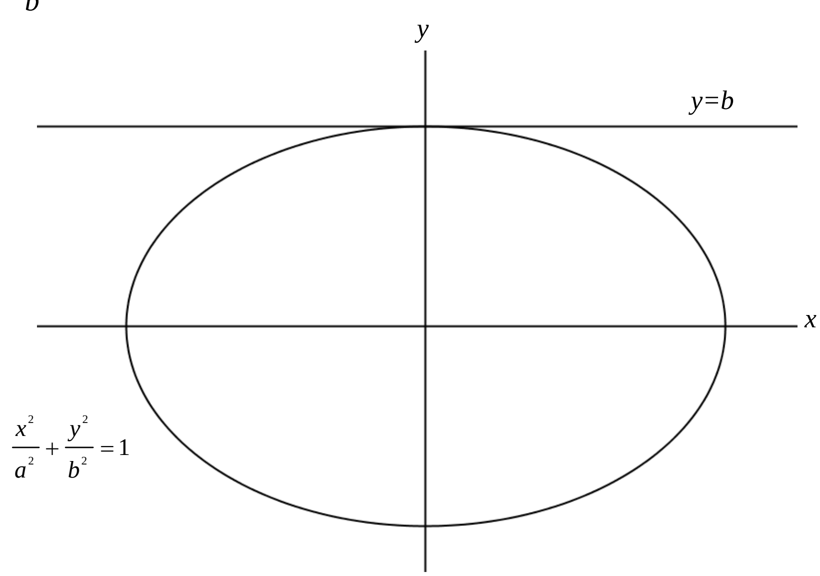
(2) $\alpha^2+\beta^2+\gamma^2$

(3) $\alpha^3+\beta^3+\gamma^3$

(ii) Hence solve simultaneously **4**

$$X+Y+Z=-1$$
$$X^2+Y^2+Z^2=5$$
$$X^3+Y^3+Z^3=-7$$

(c) The ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ is rotated about the line $y=b$. **3**

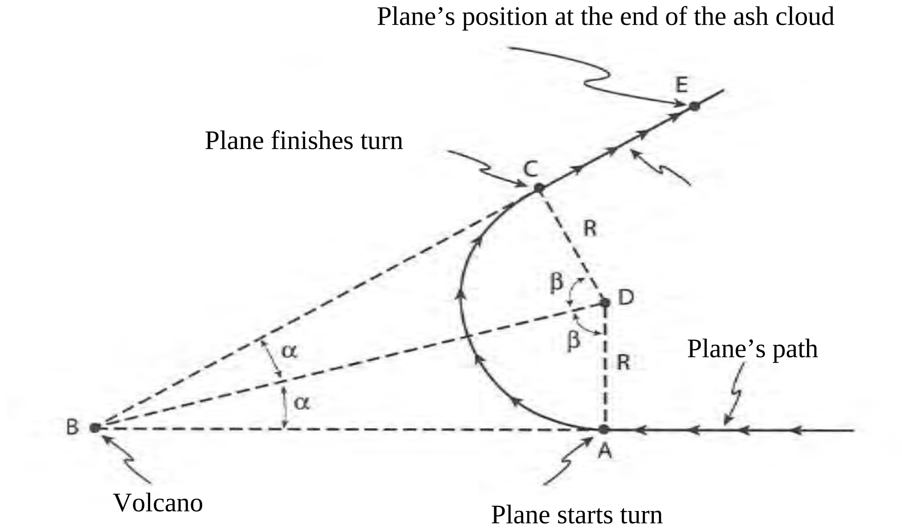


By taking slices perpendicular to the axis of rotation show that the volume, *V*, of the solid formed is

$$V=2\pi^2ab^2 \quad \text{units}^3.$$

Question 15. (15 marks) Use a *separate* writing booklet.

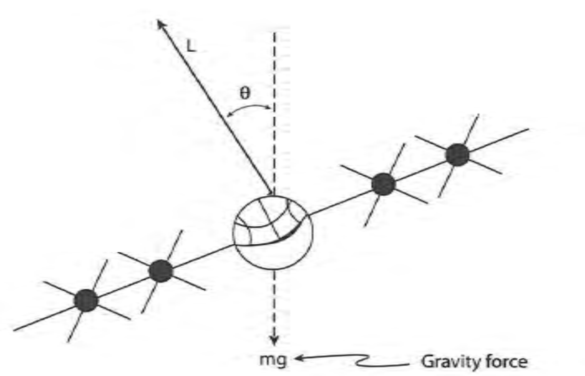
(a) An aeroplane of mass, m, is flying with velocity, $v\ ms^{-1}$, when the pilot is informed that she is about to fly directly into the path of an erupting volcano at a horizontal distance of 5970 metres from her current position, A. The pilot makes an emergency turn, at A, while maintaining the same altitude and velocity. After making the turn and heading away from the volcano, the plane travels through a volcanic ash cloud from points C to E.



(i) Show that the turn angle, 2β, for the plane is given by $2\beta = \pi - 2\tan^{-1}\left(\frac{R}{5970}\right)$ **1**

where R is the radius of the arc the plane travels along.

(ii) The forces acting on the plane while it makes its turn are gravity, mg, and the upward lift force, L, perpendicular to the wings as shown. **3**



For a semi-vertical angle of θ and velocity v, show that the radius of the turning circle, R

$$R = \frac{v^2}{g\tan\theta}$$

(iii) It is known that the semi-vertical angle, $\theta = \frac{\pi}{3}$, the plane's velocity, $v = 157 \ ms^{-1}$ and the acceleration due to gravity is $9 \cdot 8ms^{-2}$. **2**

Assuming no air resistance and uniform velocity, find the time the plane takes to complete the turn *AC*.

Correct your answer to the nearest one-tenth of a second.

(iv) As the plane comes out of the turn, point *C*, it experiences air resistance due to a volcanic ash cloud. The resistance has magnitude $mk\left(v+v^3\right)$ Newtons. **3**

If the plane heads straight for point *E* at the same altitude, show that the distance *CE* is given by

$$x = \frac{1}{k}\left(\tan^{-1}V - \tan^{-1}v\right),$$

where *V* is the velocity at point *C*.

(b) ω is a complex root of the equation $x^3 = 1$.

(i) Show that ω^2 is also a root of the equation. **1**

(ii) Show that $1+\omega+\omega^2 = 0$ and $1+\omega^2+\omega^4 = 0$ **2**

(iii) Let α and β be real numbers. Find a simplified monic cubic equation whose roots are **3**

$$\alpha+\beta,\ \alpha\omega+\beta\omega^{-1},\ \alpha\omega^2+\beta\omega^{-2}$$

Question 16. (15 marks) Use a *separate* writing booklet.

(a) Use the substitution $x = 2\sin\theta$ to calculate $\int_{-1}^{\sqrt{3}} \frac{x^2}{\sqrt{4-x^2}}\, dx$ **4**

(b) (i) Show $\frac{1-x^2}{1-x} = 1+x$ and $\frac{1-x^3}{1-x} = 1+x+x^2$ **1**

(ii) Assume $\frac{1-x^n}{1-x} = 1+x+x^2+....+x^{n-1}$. You do not need to prove this. **3**

If $I_n = \int n+(n-1)x+(n-2)x^2+...+x^{n-1}\ dx$ show that

$$I_n = \int \frac{n-(n+1)x+x^{n+1}}{(1-x)^2}\, dx\,.$$

where *n* is a positive integer.

(iii) By considering the graph of $f(x) = \frac{x^{n+1}}{(1-x)^2}$ for $0 \le x \le \frac{1}{2}$ show that **2**

$$0 \le \int_0^{\frac{1}{2}} \frac{x^{n+1}}{(1-x)^2} dx \ \le 2^{-n}$$

(iv) Deduce that for sufficiently large values of n, $\int_0^{\frac{1}{2}} \frac{x^{n+1}}{(1-x)^2} dx$ is approximately equal to 0. **1**

(v) Hence find an expression approximating

$$I_n = \int_0^{\frac{1}{2}} n+(n-1)x+(n-2)x^2+...+x^{n-1}\ dx$$

for large values of *n*. **4**

End of examination.

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

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2013

Trial HSC Examination

Assessment Task 3

Mathematics Extension 2

Solutions

Section I 10 marks

1 Let $z = 3 - i$.

What is an expression for $\dfrac{1}{z}$?

(C) $\dfrac{1}{z} = \dfrac{1}{3-i} \times \dfrac{3+i}{3+i}$

$= \dfrac{3+i}{9+1}$

$= \dfrac{3}{10} + \dfrac{1}{10}i$

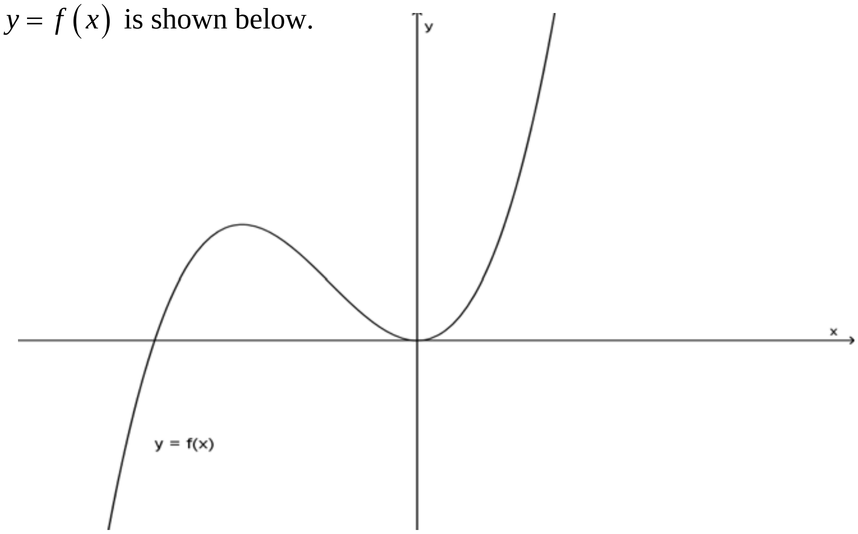
2 What are the co-ordinates for the foci of the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$

(C) Focus S $(0, \pm be)$ $e = \sqrt{1 - \left(\dfrac{2}{3}\right)^2}$

$\left(0, \pm 3\dfrac{\sqrt{5}}{3}\right)$ $= \dfrac{\sqrt{5}}{3}$

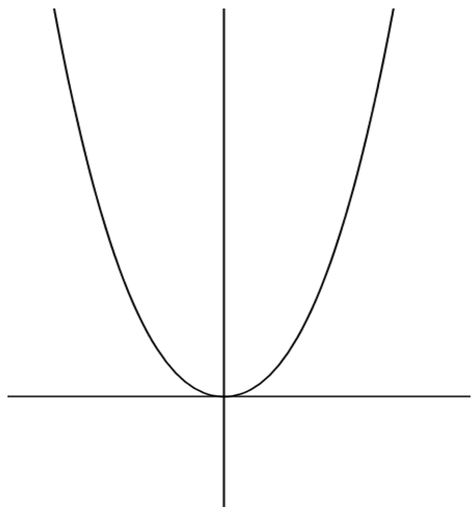
$(0, \pm\sqrt{5})$

3 The graph of $y = f\left(x\right)$ is shown below.



Which of the following graphs best represents $y = f\left(\left|x\right|\right)$?

(B)



4 Express $z = \sqrt{3} + i$ in modulus-argument form.

(C) $2\cos\frac{\pi}{6} + 2i\sin\frac{\pi}{6}$

$$\text{mod}\, z = \sqrt{\left(\sqrt{3}\right)^2 + 1^2}$$
$$= 2$$

$$\arg z = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$
$$= \frac{\pi}{6}$$

5 By using the standard table of integrals evaluate $\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{4-\sin^2 x}}\, dx$

(B) $\frac{\pi}{6}$

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{4-\sin^2 x}}\, dx = \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{2^2-(\sin x)^2}}\, dx$$

$$= \left[\sin^{-1}\left(\frac{\sin x}{2}\right)\right]_0^{\frac{\pi}{2}}$$

$$= \sin^{-1}\frac{1}{2} - \sin^{-1} 0$$

$$= \frac{\pi}{6}$$

6 What is the remainder when $x^3 + x^2 + 5x + 6$ is divided by $x + i$

(C) $5 - 4i$

$$P(-i) = (-i)^3 + (-i)^2 + 5(-i) + 6$$

$$= i - 1 - 5i + 6$$
$$= 5 - 4i$$

7 α, β, γ are the roots of $x^3 - 4x^2 + x - 5 = 0$. An equation that has roots $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ is

(B) $1 - 4x + x^2 - 5x^3 = 0$

$$P\left(\frac{1}{x}\right) = \left(\frac{1}{x}\right)^3 - 4\left(\frac{1}{x}\right)^2 + \frac{1}{x} - 5 = 0$$

$$-5x^3 + x^2 - 4x + 1 = 0$$

8 The point A represents the complex number $-4-3i$. $\angle OZA = 90^\circ$ and $|ZA| = 2|z|$.

Find the complex number represented by the point Z.

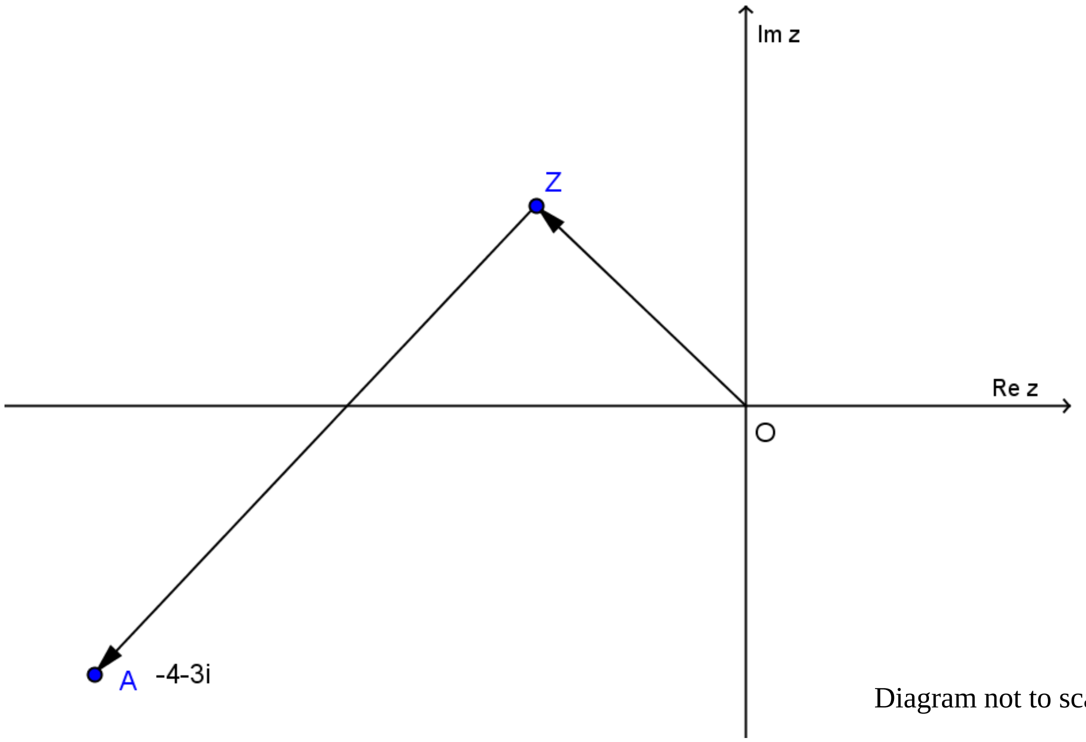


Diagram not to scale.

(C) $-2+i$

Let $z = x+iy$

$$x+iy+2i(x+iy) = -4-3i$$
$$x-2y+(2x+y)i = -4-3i$$

Comparing real & imaginary parts

$$x-2y=-4$$
$$2x+y=-3$$

$$x-2(-3-2x) = -4$$
$$x=-2$$

$$y=-3-2(-2)$$
$$=1$$

$$\therefore z=-2+i$$

9 A mass of 1 kg is released from rest at the surface of a medium in which the retardation on the mass is proportional to the distance fallen (x). The net force for this motion is $g - kx$ Newtons with the downward direction as positive.

The mass will become stationary after falling how far?

(B) $\frac{2g}{k}$

$$\ddot{x} = g - kx$$

$$\frac{1}{2}\frac{d\,v^2}{dx} = g - kx$$

$$\frac{1}{2}v^2 = gx - \frac{k}{2}x^2 + C$$

$$v = 0, x = 0 \qquad \Rightarrow\ c = 0$$

$$v^2 = 2gx - kx^2$$

$$v = 0$$

$$2gx - kx^2 = 0$$

$$x = 0 \quad or \quad x = \frac{2g}{k}$$

10 What is the approximate value of $\int_0^{\frac{\pi}{2}} \frac{1}{\cos\theta + 2\sin\theta + 3}\,d\theta$?

(A) $0\cdot322$

$$\int_0^{\frac{\pi}{2}} \frac{1}{\cos\theta + 2\sin\theta + 3}\,d\theta = \int_0^1 \frac{1}{\frac{1-t^2}{1+t^2} + 2\frac{2t}{1+t^2} + 3}\,\frac{2}{1+t^2}\,dt$$

$$= \int_0^1 \frac{2}{1-t^2+4t+3t^2+3}dt$$

$$= \int_0^1 \frac{1}{t^2+2t+2}dt$$

$$= \int_0^1 \frac{1}{1+(1+t)^2}dt$$

$$= \left[\tan^{-1}(1+t)\right]_0^1$$

$$= 0\cdot322$$

Section II

90 marks Attempt Questions 11–16 Allow about 2 hours and 45 minutes for this section

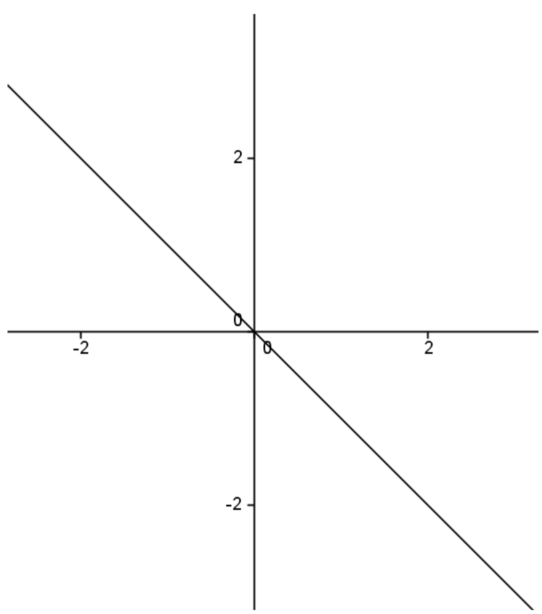
Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11. (15 marks) Use a *separate* writing booklet.

(a) On an Argand diagram, sketch the locus of the points z such that $|z+1| = |z-i|$.

Solution



Marking guideline
2 For correct solution.
1 Indicating $z - z_1$ or partial algebraic solution

Marker's Comments

Generally well done

(b) Find the square roots of $5 + 2\sqrt{6}i$.

Solution

let $z = x + iy$

$z^2 = x^2 - y^2 + 2xyi = 5 + 2\sqrt{6}i$

Comparing real & imaginary parts

Marking guideline
2 For correct solution.
1 Partial algebraic solution

$$x^2 - y^2 = 5 \qquad 2xy = 2\sqrt{6}$$

$$x^2 - \left(\frac{\sqrt{6}}{x}\right)^2 = 5$$

$$x^4 - 5x^2 - 6 = 0$$

$$\left(x^2 - 6\right)\left(x^2 + 1\right) = 0$$

$$x = \pm\sqrt{6} \quad or \quad x \neq i$$

$$y = \pm 1$$

$$z = \pm\sqrt{6} \pm i$$

Marker's Comments

Generally well done

(c) A curve is defined by the implicit equation $3x^2 + y^2 - 2xy - 8x + 2 = 0$.

(i) Show that $\frac{dy}{dx} = \frac{3x - y - 4}{x - y}$.

Solution

Deriving implicitly

$$6x + 2y\frac{dy}{dx} - 2x.1.\frac{dy}{dx} - 2y - 8 = 0$$

$$6x - 2\left(x - y\right)\frac{dy}{dx} - 2y - 8 = 0$$

$$3x - y - 4 = \left(x - y\right)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{3x - y - 4}{x - y}$$

Marking guideline

2 For correct solution.

1 Partial algebraic solution

Marker's Comments

Generally well done

(ii) Sketch the curve showing any turning points.

Solution

$$3x - y - 4 = 0$$

$$y = 3x - 4$$

$$3x^2 + \left(3x - 4\right)^2 - 2x\left(3x - 4\right) - 8x + 2 = 0$$

$$x^2 - 4x + 3 = 0$$

$$x = 3, 1$$

$$y = 5, -1$$

Marking guideline

3 For correct solution.

2 Correct shape but some incorrect values

1 Partial algebraic working

$$x \neq y$$

$$3x^2 + (x)^2 - 2x(x) - 8x + 2 = 0$$

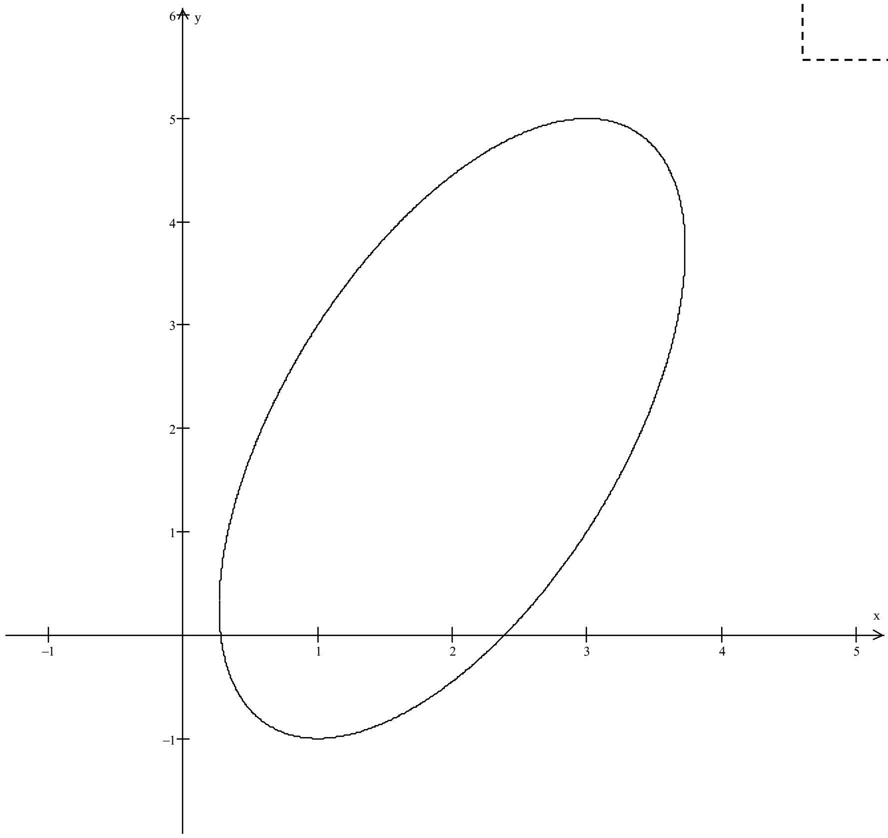
$$x^2 - 4x + 1 = 0$$

$$x = \frac{4 \pm \sqrt{12}}{2}$$

$$= 2 \pm \sqrt{3}$$

Marker's Comments

Some students had incorrect shape with correct values. These students were awarded 1 mark as per the marking guidelines.



(d) (i) Show that $z^n + z^{-n} = 2\cos n\theta$.

Solution

Let $z = \cos\theta + i\sin\theta$

$$LHS = (\cos\theta + i\sin\theta)^n + (\cos\theta + i\sin\theta)^{-n}$$

$$= \cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta)$$

$$= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$$

$$= 2\cos n\theta$$

Marking guideline

2 For correct solution.
1 Not communicating explicitly the steps

Marker's Comments

Generally well done.

(ii) Hence solve $5z^4 - z^3 + 6z^2 - z + 5 = 0$.

Solution

$$\begin{aligned}
5z^4 - z^3 + 6z^2 - z + 5 &= 0 \\
5z^2 - z + 6 - z^{-1} + 5z^{-2} &= 0 \\
5\left(z^2 + z^{-2}\right) - \left(z + z^{-1}\right) + 6 &= 0 \\
5\left(2\cos 2\theta\right) - \left(2\cos\theta\right) + 6 &= 0 \\
10\left(2\cos^2\theta - 1\right) - 2\cos\theta + 6 &= 0 \\
10\cos^2\theta - \cos\theta - 2 &= 0 \\
\left(5\cos\theta + 2\right)\left(2\cos\theta - 1\right) &= 0
\end{aligned}$$

$$\cos\theta = \frac{1}{2} \quad or \quad \cos\theta = -\frac{2}{5}$$

$$\Rightarrow \quad z = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i \qquad or \qquad z = -\frac{2}{5} \pm \frac{\sqrt{21}}{5} i$$

Marking guideline

4 For correct solution.
3 Solution contains one arithmetic error
2 Showing $10\cos^2\theta - \cos\theta - 2 = 0$
1 Showing $5z^2 - z + 6 - z^{-1} + 5z^{-2} = 0$

Marker's Comments

Not many students produced four solutions. Shows lack of understanding of the fundamental theorem of algebra.

Question 12. (15 marks) Use a *separate* writing booklet.

(a) Find $\int \frac{\sec^4\theta}{\tan\theta}\, d\theta$.

Marking guideline

3 For correct solution.

2 For finding one of the two integrals

1 For finding two simplified integrals

Solution

$$\int \frac{\sec^4\theta}{\tan\theta}\, d\theta = \int \frac{\sec^2\theta\sec^2\theta}{\tan\theta}\, d\theta$$

$$= \int \frac{\sec^2\theta}{\tan\theta}\left(1+\tan^2\theta\right) d\theta$$

$$= \int \frac{\sec^2\theta}{\tan\theta}\, d\theta + \int \sec^2\theta\tan\theta\, d\theta$$

$$= \ln\left|\tan\theta\right| + \frac{1}{2}\tan^2\theta + C$$

Marker's Comments

There were some good alternate methods.

Many careless errors around the signs.

(b) Let $I_n = \int \left(x^2+a^2\right)^n\, dx$.

(i) Show that $I_n = \frac{x\left(x^2+a^2\right)^n}{2n+1} + \frac{2na^2}{2n+1} I_{n-1}$.

Marking guideline

3 For correct solution.

2 For finding

$x\left(x^2+a^2\right)^n - 2n\int \left(x^2+a^2\right)\left(x^2+a^2\right)^{n-1} - a^2\left(x^2+a^2\right)^{n-1}\, dx$

1 For finding $x\left(x^2+a^2\right)^n - 2n\int x^2\left(x^2+a^2\right)^{n-1}\, dx$

$$I_n = \int \left(x^2+a^2\right)^n\, dx$$

$$= x\left(x^2+a^2\right)^n - \int x \times n\left(x^2+a^2\right)^{n-1} 2x\, dx$$

$$= x\left(x^2+a^2\right)^n - 2n\int x^2\left(x^2+a^2\right)^{n-1}\, dx$$

$$= x\left(x^2+a^2\right)^n - 2n\int \left(x^2+a^2-a^2\right)\left(x^2+a^2\right)^{n-1}\, dx$$

$$= x\left(x^2+a^2\right)^n - 2n\int \left(x^2+a^2\right)\left(x^2+a^2\right)^{n-1} - a^2\left(x^2+a^2\right)^{n-1}\, dx$$

$$= x\left(x^2+a^2\right)^n - 2nI_n + 2na^2\int \left(x^2+a^2\right)^{n-1}\, dx$$

$$2nI_n + I_n = x\left(x^2+a^2\right)^n + 2na^2 I_{n-1}$$

$$I_n\left(2n+1\right) = x\left(x^2+a^2\right)^n + 2na^2 I_{n-1}$$

$$I_n = \frac{x\left(x^2+a^2\right)^n}{2n+1} + \frac{2na^2}{2n+1} I_{n-1}$$

Marker's Comments.

Many students got stuck after integrating by parts and did not know how to proceed.

Some students overly complicated the question.

(ii) Hence evaluate $\int_0^1 \left(x^2+2\right)^3\,dx$.

Solution

Marking guideline
2 For correct solution
1 For solution with errors but correct procedure

$$I_3=\left[\frac{x\left(x^2+2\right)^3}{2(3)+1}\right]_0^1+\frac{2(3)2}{2(3)+1}I_{3-1}$$

$$=\left(\frac{1(1+2)^3}{7}\right)-\left(\frac{0(0+1)^3}{7}\right)+\frac{12}{7}I_2$$

$$=\frac{27}{7}+\frac{12}{7}\left[\left[\frac{x\left(x^2+2\right)^2}{2(2)+1}\right]_0^1+\frac{2(2)2}{2(2)+1}I_{2-1}\right]$$

$$=\frac{27}{7}+\frac{12}{7}\left[\left[\left(\frac{1(1+2)^2}{5}\right)-\left(\frac{0(0+1)^2}{5}\right)\right]+\frac{8}{5}I_1\right]$$

$$=\frac{27}{7}+\frac{12}{7}\left[\frac{9}{5}+\frac{8}{5}I_1\right]$$

$$=\frac{27}{7}+\frac{108}{35}+\frac{96}{35}\left[I_1\right]$$

$$=\frac{27}{7}+\frac{108}{35}+\frac{96}{35}\left[\left[\frac{x\left(x^2+2\right)^1}{2(1)+1}\right]_0^1+\frac{2(1)2}{2(1)+1}I_{1-1}\right]$$

$$=\frac{27}{7}+\frac{108}{35}+\frac{96}{35}\left[\left[\left(\frac{1(1+2)^1}{3}\right)-\left(\frac{0(0+1)^1}{3}\right)\right]+\frac{4}{3}I_0\right]$$

$$=\frac{27}{7}+\frac{72}{35}+\frac{96}{35}\left[\frac{3}{3}+\frac{4}{3}\times 1\right] \quad \text{since } I_0=\int_0^1 1\,dx=1$$

$$=\frac{467}{35}$$

Marker's Comments

Many errors included

* poor arithmetic.

* not noting $a^2 = 2$

* not evaluating between 0 & 1

Note: Since the question said hence you were expected to evaluate the integral using the recurrence formula and not by firstly expanding.

(c) An ellipse has the equation $x^2+16y^2=25$.

(i) Find the gradient of the ellipse at point $P(3,1)$.

Marking guideline
2 For correct solution
1 For solution with errors but correct procedure

$$2x+32y\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{x}{16y}$$

at $P(3,1)$ $\quad \frac{dy}{dx}=-\frac{3}{16}$

Marker's Comments

When deriving implicitly, the derivative of 25 is 0 & not 25.

Too many carless errors.

(ii) Find the equation of the tangent and normal to the ellipse at P.

Solution

Tangent:

$$y-1=\frac{-3}{16}(x-3)$$

$$3x+16y-25=0$$

Normal:

$$y-1=\frac{16}{3}(x-3)$$

$$16x-3y-45=0$$

Marking guideline

2 For correct solution

1 For one correct solution

Marker's Comments

Generally well answered

(iii) The normal to the ellipse, at point P, meets the major axis at Q. A line from the centre, O to the tangent at P meets at right angles at point A. Show that the value of $PQ\times OA$ is equal to the square of the semi-minor axis.

Solution

Semi-minor axis: $\frac{x^2}{25}+\frac{16y^2}{25}=1 \quad \Rightarrow \quad b=\frac{\sqrt{25}}{\sqrt{16}} \quad b^2=\frac{25}{16}$

$$OA=\frac{|3\times 0+4\times 0-25|}{\sqrt{3^2+16^2}}$$

$$=\frac{25}{\sqrt{265}}$$

PQ hits A at $y=0 \quad \Rightarrow \quad x=\frac{45}{16}$

$$PQ^2=\left(3-\frac{45}{16}\right)^2+(1-0)^2$$

$$PQ=\frac{\sqrt{265}}{16}$$

$$PQ\times OA=\frac{\sqrt{265}}{16}\times\frac{25}{\sqrt{265}}$$

$$=\frac{25}{16}$$

$$=b^2$$

Marking guideline

3 For correct solution including an explanation of b^2.

2 For finding 2 parts (OA, PQ or b^2)

1 For finding 1 part (OA, PQ or b^2)

Marker's Comments

Students need to show $b^2 = 1.5625$ & not just assume $PQ\times OA = b^2$.

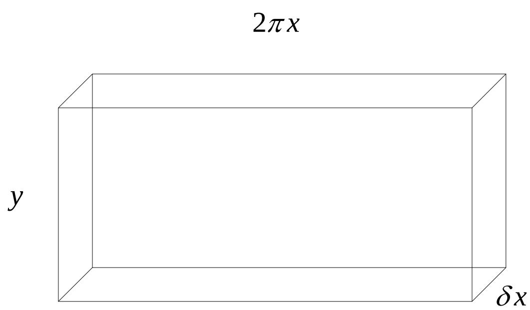
Some students took the longer method of using the distance formula to find OA rather than using the perpendicular distance.

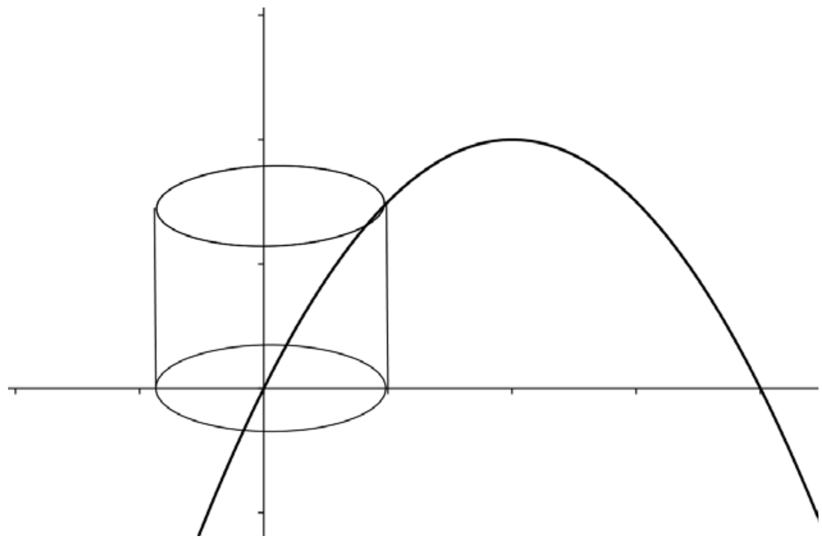
Question 13. (15 marks) Use a *separate* writing booklet.

(a) The area bounded by the curve $y = x(2-x)$ and the x-axis is rotated about the y-axis.

(i) By using the method of cylindrical shells show that the volume, $V = 2\pi\int_0^2 xy \; dx$

Solution





$$\delta V = 2\pi x y \delta x$$

$$V = \lim_{\delta x \to 0}\sum_0^2 2\pi x y \delta x$$
$$= 2\pi\int_0^2 xy \; dx$$

Marking guideline
2 For correct solution
1 For shell diagram and derivation of δV

Marker's Comments
Many students did not draw a diagram

Links between steps were often not clearly shown.

(ii) Hence find the volume of the solid.

Solution

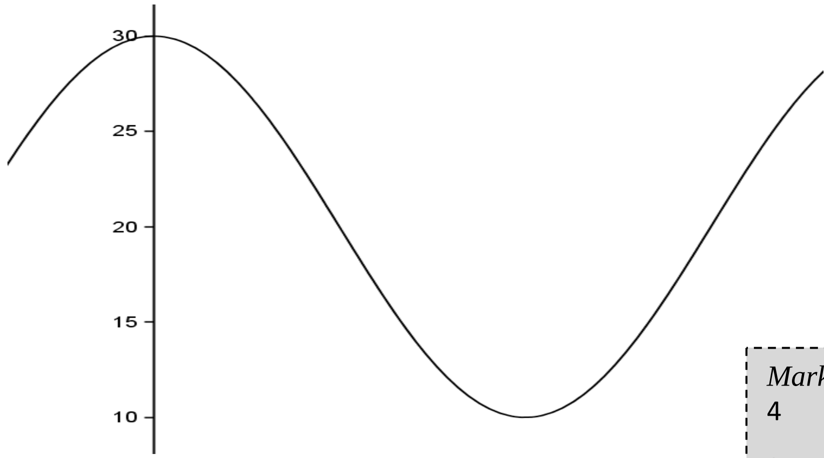
$$V = 2\pi\int_0^2 xy \; dx$$
$$= 2\pi\int_0^2 x\left[x(2-x)\right] dx$$
$$= 2\pi\int_0^2 2x^2 - x^3 \; dx$$
$$= 2\pi\left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^2$$
$$= \frac{8\pi}{3}$$

Marking guideline
3 For correct solution
2 For finding $2\pi\left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^2$
1 For finding $2\pi\int_0^2 2x^2 - x^3 \; dx$

Marker's Comments
Well answered

(b) High tide for a harbour occurs at 5 a.m. and low tide at 11:20 a.m. with the depths of the harbour at high and low tide 30 and 10 metres respectively. If the tidal motion is assumed to be in simple harmonic motion find the latest time before noon that a ship needing 25 metres clearance could exit the harbour.

Solution



Marking guideline

4 For correct solution

3 For finding $t = \frac{380}{3}$

2 For finding $x = 10\cos\frac{\pi}{380}t + 20$

1 for finding *a* or *n* correctly

High tide to Low tide = 380 mins $\Rightarrow$ T = 760 mins

Now SHM $\Rightarrow$ $x = a\cos nt + 20$

$$= 10\cos\frac{\pi}{380}t + 20 \qquad \textit{as} \quad a = 10 \quad n = \frac{2\pi}{760}$$

When $x = 25\ m$ $\quad 25 = 10\cos\frac{\pi}{380}t + 20$

$$\frac{\pi}{380}t = \frac{\pi}{3}, \frac{5\pi}{3}, \ldots$$

$$t = \frac{380}{3}, \ldots$$

$\therefore$ the latest it could leave is 7:07 am to the nearest minute.

Marker's Comments

Many students did not draw a diagram to assist understanding

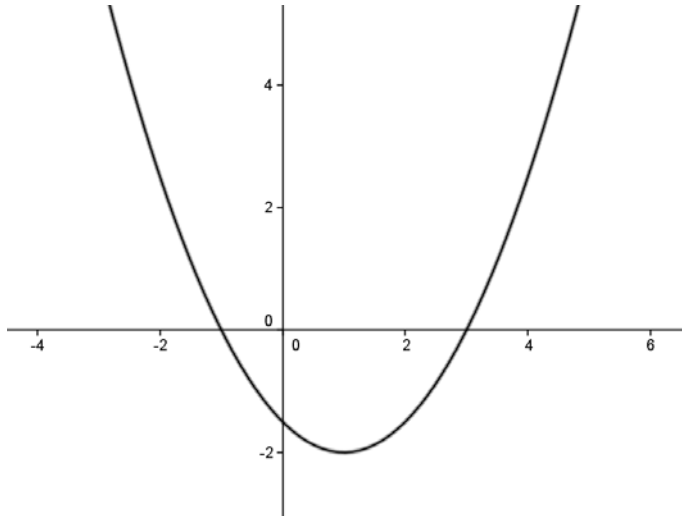
Students need to be clear on the units for t

Many students did not state the amplitude which would have earned them a mark

The best solutions let t = 0 for 5 am and had the form $x = a\cos nt + C$

Many students did not identify the centre of motion.

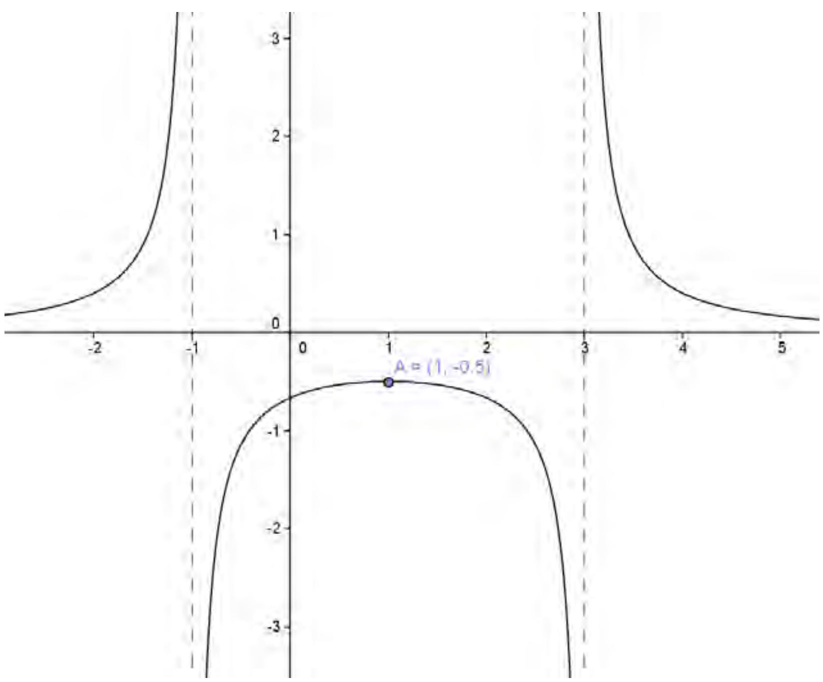
(c) The diagram shows the graph of $f\left(x\right)=\frac{1}{2}\left(x-3\right)\left(x+1\right)$.



Use the above graph to draw a one-third page sketch, showing any intercepts or asymptotes, of

(i) $y=\frac{1}{f\left(x\right)}$ (Show the coordinates of the turning point)

Solution



Marking guideline

3 For correct solution

2 No indication of turning point or labels or asymptotes

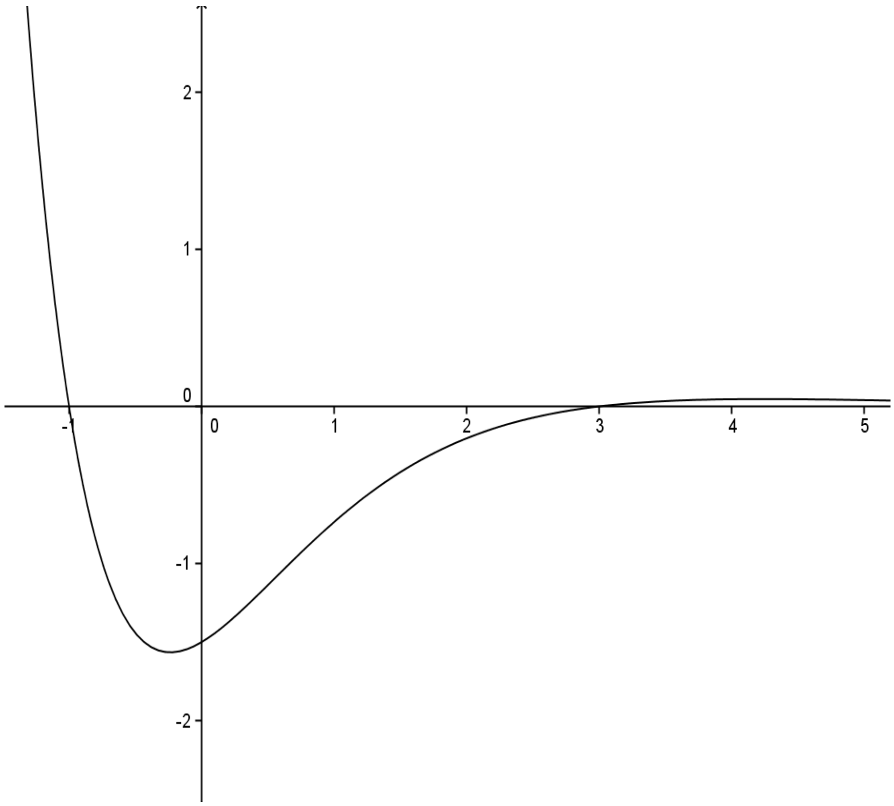
1 Some correct features

Marker's Comments

Generally marks were lost for not being clear about asymptotes, asymptotic behaviour, intercepts, turning points etc.

(ii) $y = e^{-x} f(x)$ (Do **NOT** show the coordinates of the turning point)

Solution



Marking guideline

3 For correct solution

2 Incorrect approximation of turning points or intercepts

1 Some correct features

Marker's Comments

Student need to label working sketches i.e. $y = e^{-x}$. It was difficult to distinguish between the solution and the preliminary sketches.

The turning point (while the coordinates were not required) needed to be shown to the left of the y-axis.

Question 14. (15 marks) Use a *separate* writing booklet.

(a) (i) Find real numbers a, b and c such that $\dfrac{1}{x^3-1}=\dfrac{a}{x-1}+\dfrac{bx+c}{x^2+x+1}$.

Solution

$$\frac{1}{x^3-1}=\frac{a}{x-1}+\frac{bx+c}{x^2+x+1}$$

$$1=a\left(x^2+x+1\right)+\left(bx+c\right)\left(x-1\right)$$

When $x=1$ $a=\dfrac{1}{3}$

Expanding & comparing coefficients

$b=-\dfrac{1}{3}$, $c=-\dfrac{2}{3}$

Marking guideline

2 For correct solution

1 For partially correct solution

Marker's Comments

(ii) Hence find $\displaystyle\int\frac{dx}{x^3-1}$.

Solution

$$\int\frac{dx}{x^3-1}=\int\frac{1}{3(x-1)}-\frac{x+2}{3\left(x^2+x+1\right)}dx$$

$$=\frac{1}{3}\int\frac{1}{x-1}\,dx-\frac{1}{3}\int\frac{x+2}{x^2+x+1}\,dx$$

$$=\frac{1}{3}\ln\left(x-1\right)-\frac{1}{3}\times\frac{1}{2}\int\frac{2x+4}{x^2+x+1}dx$$

$$=\frac{1}{3}\ln\left(x-1\right)-\frac{1}{6}\int\frac{(2x+1)\ +\ 3}{x^2+x+1}dx$$

$$=\frac{1}{3}\ln\left(x-1\right)-\frac{1}{6}\int\frac{2x+1}{x^2+x+1}dx-\frac{1}{6}\int\frac{3}{x^2+x+1}\,dx$$

$$=\frac{1}{3}\ln\left(x-1\right)-\frac{1}{6}\ln\left(x^2+x+1\right)-\frac{1}{2}\int\frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx$$

$$=\frac{1}{3}\ln\left(x-1\right)-\frac{1}{6}\ln\left(x^2+x+1\right)-\frac{1}{2}\left[\frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\left(\frac{x+1}{\frac{\sqrt{3}}{2}}\right)\right]$$

$$=\frac{1}{3}\ln\left(x-1\right)-\frac{1}{6}\ln\left(x^2+x+1\right)-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\left(x+1\right)}{\sqrt{3}}\right)+C$$

Marking guideline

3 For correct solution

2 For finding $\displaystyle\int\frac{1}{x-1}dx=\ln\left(x-1\right)$ and $\displaystyle\int\frac{2x+1}{x^2+x+1}dx$

1 For finding $\displaystyle\int\frac{1}{x-1}dx=\ln\left(x-1\right)$

Marker's Comments

Students need to take greater care with the signs & simplifying terms.

(b) α, β, γ are the roots of the equation $x^3 - px^2 + qx - r = 0$.

(i) Find an expression for the following in terms of *p*,*q*,*r*.

(1) $\alpha + b + \gamma$

(2) $\alpha^2 + \beta^2 + \gamma^2$

(3) $\alpha^3 + \beta^3 + \gamma^3$

Marking guideline
3 For correct solution
2 For two correct parts
1 For one correct part

Solution

(1) $\alpha + \beta + \gamma = p$

$$note \;\; \alpha\beta + \alpha\gamma + \beta\gamma = q$$
$$\alpha\beta\gamma = r$$

Marker's Comments

(2) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$
$$= p^2 - 2q$$

(3) $\alpha^3 - p\alpha^2 + q\alpha - r = 0$
$$\beta^3 - p\beta^2 + q\beta - r = 0$$
$$\gamma^3 - p\gamma^2 + q\gamma - r = 0$$

adding $\alpha^3 + \beta^3 + \gamma^3 - p(\alpha^2 + \beta^2 + \gamma^2) + q(\alpha + \beta + \gamma) - 3r = 0$
$$\alpha^3 + \beta^3 + \gamma^3 - p(p^2 - 2q) + q(p) - 3r = 0$$
$$\alpha^3 + \beta^3 + \gamma^3 = p^3 - 3pq + 3r$$

(ii) Hence solve simultaneously

$$X + Y + Z = -1$$
$$X^2 + Y^2 + Z^2 = 5$$
$$X^3 + Y^3 + Z^3 = -7$$

Marking guideline
3 For correct solution
2 For two correct parts
1 For one correct part

Solution

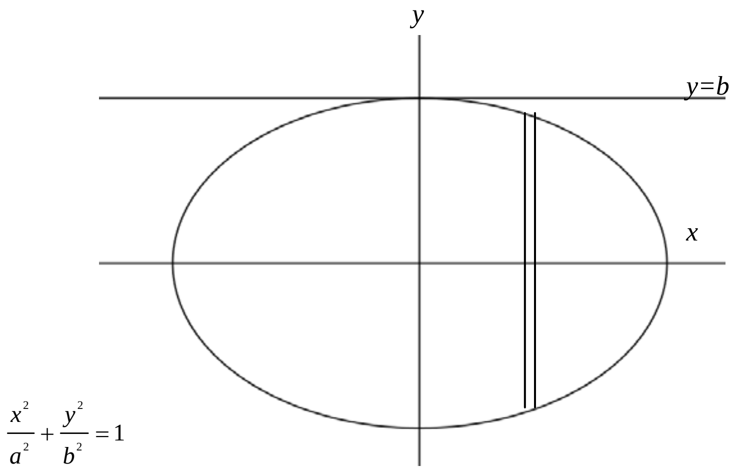
Let $X = \alpha, Y = \beta$ & $Z = \gamma$

$$\alpha + \beta + \gamma = -1 \quad \Rightarrow \quad p = -1$$
$$\alpha^2 + \beta^2 + \gamma^2 = 5 \quad p^2 - 2q = 5 \Rightarrow q = -2$$
$$\alpha^3 + \beta^3 + \gamma^3 = -7 \quad p^3 - 3pq + 3r = -7 \Rightarrow r = 0$$

Marker's Comments

Therefore *X*,*Y* and *Z* are solutions to $x^3 + x^2 - 2x = 0$
$$x(x^2 + x - 2) = 0$$
$$X = 0, Y = 1, \; Z = -2$$

(c) The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is rotated about the line $y=b$.



By taking slices perpendicular to the axis of rotation show that the volume, *V*, of the solid formed is

$$V=2\pi^2ab^2 \quad \text{units}^3.$$

Solution

$$\begin{aligned}\delta V &= \pi\left(\left(b+y\right)^2-\left(b-y\right)^2\right)\delta x\\ &= \pi\left(b^2+2by+y^2-b^2+2by-y^2\right)\delta x\\ &= 4\pi by\delta x\\ &= 4\pi b\sqrt{b^2-\frac{b^2}{a^2}x^2}\delta x\\ &= 4\pi b^2\sqrt{1-\frac{x^2}{a^2}}\delta x\\ &= 4\pi\frac{b^2}{a}\sqrt{a^2-x^2}\delta x\end{aligned}$$

$$\begin{aligned}V &= \lim_{\delta x\to 0}\sum_{-a}^{a}4\pi\frac{b^2}{a}\sqrt{a^2-x^2}\delta x\\ &= 4\pi\frac{b^2}{a}\int_{-a}^{a}\sqrt{a^2-x^2}dx\\ &= 4\pi\frac{b^2}{a}\left[\frac{1}{2}\pi a^2\right]\\ &= 2\pi^2ab^2\end{aligned}$$

Marking guideline

3 For correct solution

2 For showing V= $4\pi\frac{b^2}{a}\int_{-a}^{a}\sqrt{a^2-x^2}dx$

1 For showing $\delta V=4\pi by\delta x$

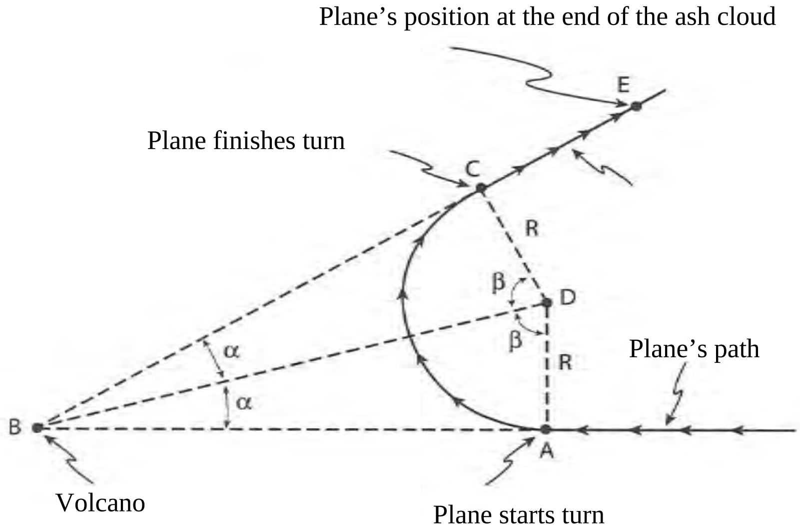
Marker's Comments

Many students found it difficult to determine the radii of the annulus.

Some students complicated the algebra by substituting an expression for y to early.

Question 15. (15 marks) Use a *separate* writing booklet.

(a) An aeroplane of mass, m, is flying with velocity, $v\ ms^{-1}$, when the pilot is informed that she is about to fly directly into the path of an erupting volcano at a horizontal distance of 5970 metres from her current position, A. The pilot makes an emergency turn, at A, while maintaining the same altitude and velocity. After making the turn and heading away from the volcano, the plane travels through a volcanic ash cloud from points C to E.



(i) Show that the turn angle, 2β, for the plane is given by $2\beta = \pi - 2\tan^{-1}\left(\frac{R}{5970}\right)$

where R is the radius of the arc the plane travels along.

Solution

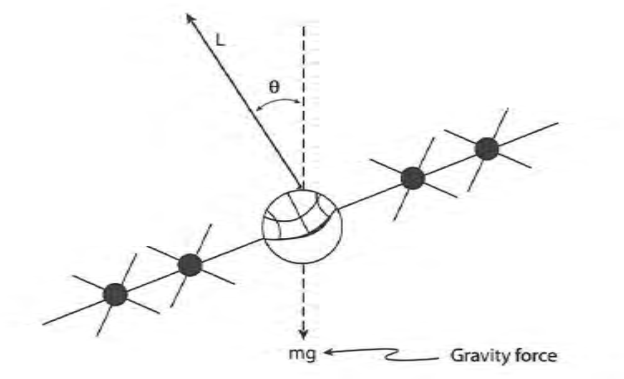
From the diagram
$$\tan\alpha = \frac{R}{5970}$$
$$\alpha = \tan^{-1}\frac{R}{5970}$$
$$\beta = \frac{\pi}{2} - \alpha$$
$$2\beta = \pi - 2\tan^{-1}\left(\frac{R}{5970}\right)$$

Marking guideline
1 For correct solution

Marker's Comments

Surprisingly this question was poorly answered by many students.

(ii) The forces acting on the plane while it makes its turn are gravity, mg, and the upward lift force, L, perpendicular to the wings as shown.



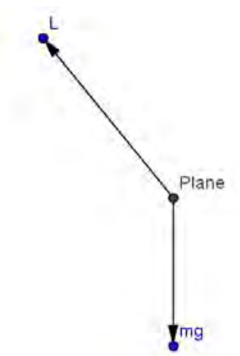
For a semi-vertical angle of θ and velocity v, show that the radius of the turning circle, R

$$R = \frac{v^2}{g\tan\theta}$$

Solution

Horizontal: $L\sin\theta = m\frac{v^2}{R}$

Vertical: $L\cos\theta = mg$



$$\frac{L\sin\theta}{L\cos\theta} = \frac{\frac{mv^2}{R}}{mg}$$

$$\tan\theta = \frac{v^2}{Rg}$$

$$R = \frac{v^2}{g\tan\theta}$$

Marking guideline

3	For correct solution
2	For solution but not explicit in all steps
1	For showing Horizontal/Vertical forces

Marker's Comments

Generally well answered.

(iii) It is known that the semi-vertical angle, $\theta = \frac{\pi}{3}$, the plane's velocity, $v = 157\ ms^{-1}$ and the acceleration due to gravity is $9 \cdot 8ms^{-2}$.

Assuming no air resistance and uniform velocity, find the time the plane takes to complete the turn *AC*.

Correct your answer to the nearest one-tenth of a second.

Solution

$$T = \frac{AC}{157}$$

$$= R \times 2\beta \qquad \textit{Note}: \text{ arc length } l = r\theta$$

$$= \frac{v^2}{g\tan\theta}\left(\pi - 2\tan^{-1}\left(\frac{\frac{v^2}{g\tan\theta}}{5970}\right)\right) \div 157$$

$$= \frac{157^2}{9 \cdot 8\tan\frac{\pi}{3}}\left(\pi - 2\tan^{-1}\left(\frac{\frac{157^2}{9 \cdot 8\tan\frac{\pi}{3}}}{5970}\right)\right) \div 157$$

$$= 24 \cdot 6 \text{ sec}$$

Marking guideline
2 For correct solution
1 For partially correct solution

Marker's Comments

Many students
* could not calculate arc length.
* did not consider the accuracy of their answer – one student wrote 2 million seconds which is approximately 550 hours.
* some students mixed up the concept of degrees & radians

(iv) As the plane comes out of the turn, point *C*, it experiences air resistance due to a volcanic ash cloud. The resistance has magnitude $mk\left(v + v^3\right)$ Newtons.

If the plane heads straight for point *E* at the same altitude, show that the distance *CE* is given by

$$x = \frac{1}{k}\left(\tan^{-1}V - \tan^{-1}v\right),$$

where *V* is the velocity at point *C*.

Solution

$$\ddot{x} = -k(v + v^3)$$

$$v\frac{dv}{dx} = -k(v + v^3)$$

$$v\frac{dv}{-kdx} = v + v^3$$

$$\frac{dv}{-kdx} = 1 + v^2$$

Marking guideline
3 For correct solution
2 For solution but not explicit in all steps
1 For showing $v\frac{dv}{-kdx} = v + v^3$

$$-k\frac{dx}{dv} = \frac{1}{1+v^2}$$

$$-k\ x = \tan^{-1}v + C$$

When $x = 0$, $v = V$

$$-k\,(0) = \tan^{-1}V + C$$

$$C = -\tan^{-1}V$$

$$x = \frac{-1}{k}(\tan^{-1}v - \tan^{-1}V)$$

$$x = \frac{1}{k}\left(\tan^{-1}V - \tan^{-1}v\right)$$

Marker's Comments

Generally answered very well.

(b) ω is a complex root of the equation $x^3 = 1$.

(i) Show that ω^2 is also a root of the equation.

Solution

If ω is a root then $\omega^3 = 1$

Now $P\left(\omega^2\right) = \left(\omega^2\right)^3 = 1$

$$\left(\omega^3\right)^2 = 1$$

$$(1)^2 = 1 \quad \textit{True}$$

Marking guideline

1 For correct solution

Marker's Comments

Many students did not consider the factor theorem.
Note $w \neq 1$ as w is a complex root
Some students found the roots to show w^2 –time wise this is a longer method.

(ii) Show that $1+\omega+\omega^2 = 0$ and $1+\omega^2+\omega^4 = 0$

Solution

Since ω is a root then

$$\omega^3 - 1 = 0$$

$$(\omega-1)\left(w^2+w+1\right) = 0$$

$\Rightarrow \omega = 1$ Not true since ω is an unreal root or

$$\omega^2 + \omega + 1 = 0$$

$$1+\omega^2+\omega^4 = 1+\omega^2+\omega\times\omega^3$$
$$= 1+\omega^2+\omega \quad \text{since } \omega^3 = 1$$
$$= 0$$

Marking guideline

2 For correct solution

1 For showing one solution

Marker's Comments

(iii) Let α and β be real numbers. Find a simplified monic cubic equation whose roots are

$$\alpha+\beta,\ \alpha\omega+\beta\omega^{-1},\ \alpha\omega^2+\beta\omega^{-2}$$

$$\sum\alpha: \quad \alpha+\beta \ + \ \alpha\omega+\beta\omega^{-1} \ + \ \alpha\omega^2+\beta\omega^{-2}=-\frac{b}{a}$$

$$\alpha\left(1+\omega+\omega^2\right)+\beta\left(1+\omega^{-1}+\omega^{-2}\right)=-b$$

$$\alpha(0)+\beta\left(\frac{\omega^2+\omega+1}{\omega^2}\right)=-b$$

$$b=0$$

Marking guideline

3 For correct solution

2 For finding two coefficients

1 For finding one coefficient

$$\sum\alpha\beta: \quad (\alpha+\beta)\left(\alpha\omega+\beta\omega^{-1}\right)+(\alpha+\beta)\left(\alpha\omega^2+\beta\omega^{-2}\right)+\left(\alpha\omega+\beta\omega^{-1}\right)\left(\alpha\omega^2+\beta\omega^{-2}\right)=\frac{c}{a}$$

$$\alpha^2\omega+\alpha\beta\omega+\alpha\beta\omega^{-1}+\beta^2\omega^{-1} \ + \ \alpha^2\omega^2+\alpha\beta\omega^{-2}+\alpha\beta\omega^2+\beta^2\omega^{-2} \ + \ \alpha^2\omega^3+\alpha\beta\omega^{-1}+\alpha\beta\omega+\beta^2\omega^{-3}=c$$

$$\alpha^2\left(\omega+\omega^2+\omega^3\right) \ + \ \alpha\beta\left(\omega+\omega^{-1}+\omega^2+\omega^{-2}+\omega+\omega^{-1}\right) \ + \ \beta^2\left(\omega^{-1}+\omega^{-2}+\omega^{-3}\right)=c$$

$$\alpha^2\left([\omega+\omega^2]+\omega^3\right) \ + \ \alpha\beta\left(\frac{\omega^3+\omega+\omega^4+1+\omega^3+\omega}{\omega^2}\right) \ + \ \beta^2\left(\frac{\omega^2+\omega+1}{\omega^3}\right)=c$$

$$\alpha^2(-1+1) \ + \ \alpha\beta\left(\frac{1+\omega+\omega.\omega^3+1+1+\omega}{-1(1+\omega)}\right) \ + \ \beta^2(0)=c$$

$$-\alpha\beta\left(\frac{3+3\omega}{1+\omega}\right)=c$$

$$c=-3\alpha\beta$$

$$\sum\alpha\beta\gamma: \quad (\alpha+\beta)\left(\alpha\omega+\beta\omega^{-1}\right)\left(\alpha\omega^2+\beta\omega^{-2}\right)=-\frac{d}{a}$$

$$(\alpha+\beta)\left(\alpha^2\omega^3+\alpha\beta\omega^{-1}+\alpha\beta\omega+\beta^2\omega^{-3}\right)=-d$$

$$(\alpha+\beta)\left(\alpha^2+\alpha\beta\left(\omega+\omega^{-1}\right)+\beta^2\right)=-d$$

$$(\alpha+\beta)\left(\alpha^2+\alpha\beta\left(\omega+\omega^{-1}\right)+\beta^2\right)=-d$$

$$\alpha^3+\alpha^2\beta\left(\omega+\omega^{-1}\right)+\alpha\beta^2+\alpha^2\beta+\beta^3+\alpha\beta^2\left(\omega+\omega^{-1}\right)=-d$$

$$\alpha^3+\beta^3+\alpha^2\beta\left(\omega+\omega^{-1}+1\right)+\alpha\beta^2\left(\omega+\omega^{-1}+1\right)=-d$$

$$\alpha^3+\beta^3+\alpha^2\beta\left(\frac{\omega^2+1+\omega}{\omega}\right)+\alpha\beta^2\left(\frac{\omega^2+1+\omega}{\omega}\right)=-d$$

$$d=-\alpha^3-\beta^3$$

$$\therefore \quad P(x)=x^3-3\alpha\beta x-\left(\alpha^3+\beta^3\right)$$

Marker's Comments

Many students did not make the connection with sum & product of roots.

Those that did make the connection did not simplify by substituting in the identities established above.

Question 16. (15 marks) Use a *separate* writing booklet.

(a) Use the substitution $x = 2\sin\theta$ to calculate $\int_{-1}^{\sqrt{3}} \frac{x^2}{\sqrt{4-x^2}}\,dx$

Solution

$$\int_{-1}^{\sqrt{3}} \frac{x^2}{\sqrt{4-x^2}}\,dx = \int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(2\sin\theta)^2}{\sqrt{4-(2\sin\theta)^2}}\,2\cos\theta\ d\theta$$

$$= \int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{4\sin^2\theta}{\sqrt{4(1-\sin^2\theta)}}\,2\cos\theta\ d\theta$$

$$= 4\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin^2\theta}{\sqrt{\cos^2\theta}}\cos\theta\ d\theta$$

$$= 4\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \sin^2\theta\ d\theta$$

$$= 4\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{2}-\frac{1}{2}\cos 2\theta\ d\theta$$

$$= 4\left[\frac{x}{2}-\frac{1}{4}\sin 2\theta\right]_{-\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= 4\left[\left(\frac{\pi}{6}-\frac{1}{4}\sin\frac{2\pi}{3}\right)-\left(-\frac{\pi}{12}-\frac{1}{4}\sin\frac{-\pi}{3}\right)\right]$$

$$= 4\left[\frac{\pi}{6}-\frac{\sqrt{3}}{8}+\frac{\pi}{12}-\frac{\sqrt{3}}{8}\right]$$

$$= \pi - \sqrt{3}$$

$$x = 2\sin\theta$$

$$\frac{dx}{d\theta} = 2\cos\theta$$

$$dx = 2\cos\theta\, d\theta$$

$$x = \sqrt{3} \quad \theta = \frac{\pi}{3}$$

$$x = -1 \quad \theta = -\frac{\pi}{6}$$

Marking guideline

4 For correct solution

3 For finding $4\left[\frac{x}{2}-\frac{1}{4}\sin 2\theta\right]_{-\frac{\pi}{6}}^{\frac{\pi}{3}}$

2 For finding $4\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \sin^2\theta\ d\theta$

1 For finding initial substitutions

Marker's Comments

Many students had trouble simplifying to $4\sin^2\theta$ with many errors in the denominator.

Some students had difficulty evaluating $\sin\left(-\frac{\pi}{6}\right)$

(b) (i) Show $\frac{1-x^2}{1-x}=1+x$ and $\frac{1-x^3}{1-x}=1+x+x^2$

Solution

$$LHS=\frac{1-x^2}{1-x}$$
$$=\frac{(1-x)(1+x)}{1-x}$$
$$=1+x$$

$$LHS=\frac{1-x^3}{1-x}$$
$$=\frac{(1-x)(1+x+x^2)}{1-x}$$
$$=1+x+x^2$$

Marking guideline
1 For correct solution

Marker's Comments
Generally well done

(ii) Assume $\frac{1-x^n}{1-x}=1+x+x^2+....+x^{n-1}$. You do not need to prove this.

If $I_n=\int n+(n-1)x+(n-2)x^2+...+x^{n-1}\ dx$ show that

$$I_n=\int\frac{n-(n+1)x+x^{n+1}}{(1-x)^2}\,dx\,.$$

where n is a positive integer.

Solution

$$I_n=\int n+(n-1)x+(n-2)x^2+...+x^{n-1}\ dx$$

$$=\int \quad 1 \quad + \quad 1+x \quad + \quad 1+x+x^2 \quad + \quad ... \quad + \quad 1+x+x^2+...+x^{n-1}\ dx$$

$$=\int\frac{1-x}{1-x}+\frac{1-x^2}{1-x}+\frac{1-x^3}{1-x}+...+\frac{1-x^n}{1-x}\ dx$$

$$=\int\frac{1}{1-x}\left[1-x+1-x^2+1-x^3+...+1-x^n\right]dx$$

$$=\int\frac{1}{1-x}\left[n-\left(x+x^2+x^3+...+x^n\right)\right]dx$$

$$=\int\frac{1}{1-x}\left[n-\left(\frac{x(1-x^n)}{1-x}\right)\right]dx$$

$$=\int\frac{1}{1-x}\left[\frac{n-nx-x+x^{n+1}}{1-x}\right]dx$$

$$I_n=\int\frac{n-(n+1)x+x^{n+1}}{(1-x)^2}\,dx$$

Marking guideline
3 For correct solution
2 For finding establishing a G.P
1 For using previous identities

Marker's Comments

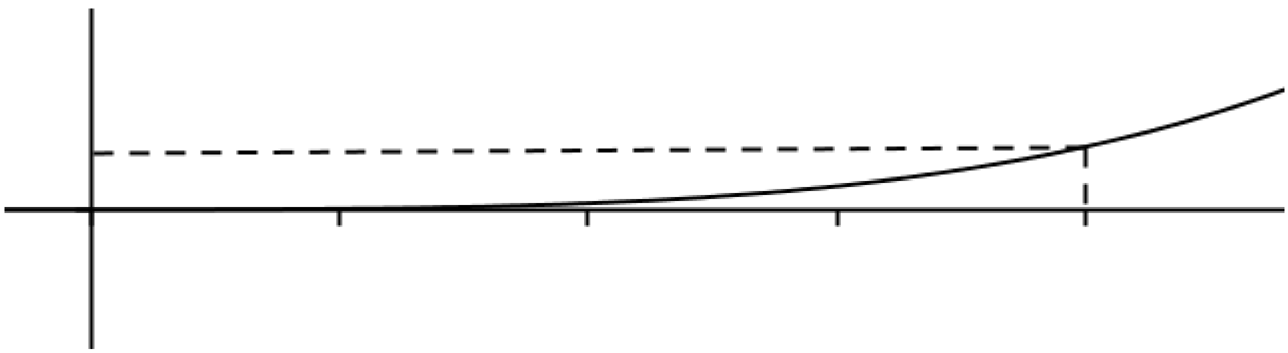
Most students had difficulty expanding and only a few were able to use the identities from part (i) to proceed any further.

(iii) By considering the graph of $f(x)=\dfrac{x^{n+1}}{(1-x)^2}$ for $0\le x\le \dfrac{1}{2}$ show that

$$0\le \int_0^{\frac{1}{2}} \frac{x^{n+1}}{(1-x)^2}\,dx \le 2^{-n}$$

Solution

The graph of $f(x)=\dfrac{x^{n+1}}{(1-x)^2}$ for $0\le x\le \dfrac{1}{2}$



$$0\le \int_0^{\frac{1}{2}} \frac{x^{n+1}}{(1-x)^2}\,dx \le \text{Area of upper rectangle}$$

$$0\le \int_0^{\frac{1}{2}} \frac{x^{n+1}}{(1-x)^2}\,dx \le \frac{1}{2}\times\frac{\left(\frac{1}{2}\right)^{n+1}}{\left(1-\frac{1}{2}\right)^2}$$

$$0\le \int_0^{\frac{1}{2}} \frac{x^{n+1}}{(1-x)^2}\,dx \le \frac{1}{2}\times\frac{\frac{1}{2^{n+1}}}{\frac{1}{4}}$$

$$0\le \int_0^{\frac{1}{2}} \frac{x^{n+1}}{(1-x)^2}\,dx \le \frac{1}{2}\times\frac{1}{2^{n+1}}\times 4$$

$$0\le \int_0^{\frac{1}{2}} \frac{x^{n+1}}{(1-x)^2}\,dx \le 2^{-n}$$

Marking guideline

2 For correct solution

1 For diagram & function value

Marker's Comments

By considering the graph of should have been a clear indicator to draw a diagram. A large number of students did not sketch.

Comparison to the upper limit given by the rectangle was poor and many students could not evaluate f(½).

(iv) Deduce that for sufficiently large values of n, $\int_0^{\frac{1}{2}} \frac{x^{n+1}}{(1-x)^2} dx$ is approximately equal to 0.

Solution

For large values of $n \Rightarrow n \to \infty$

As $n \to \infty$ $2^{-n} \to 0$

$$\Rightarrow \quad 0 \le \int_0^{\frac{1}{2}} \frac{x^{n+1}}{(1-x)^2} dx \le 0$$

$$\Rightarrow \quad \int_0^{\frac{1}{2}} \frac{x^{n+1}}{(1-x)^2} dx \approx 0$$

Marking guideline

1 For correct solution

Marker's Comments

This part references the result of a previous part which is a common thing is the HSC .

Attempts to use $x^{n+1} \to 0$ as $n \to \infty$ drew no marks.

Very poor use of limits, too many incorrectly substituted $n = \infty$.

(v) Hence find an expression approximating

$$I_n = \int_0^{\frac{1}{2}} n + (n-1)x + (n-2)x^2 + \ldots + x^{n-1} \; dx$$

for large values of n.

Solution

$$\begin{aligned} I_n &= \int_0^{\frac{1}{2}} n + (n-1)x + (n-2)x^2 + \ldots + x^{n-1} \; dx \\ &= \int \frac{n - (n+1)x + x^{n+1}}{(1-x)^2} \; dx \\ &= \int \frac{n}{(1-x)^2} \; dx - \int \frac{(n+1)x}{(1-x)^2} \; dx + \int \frac{x^{n+1}}{(1-x)^2} \; dx \end{aligned}$$

Marking guideline

4 For correct solution.

3 For only one arithmetic or algebraic error.

2 For integrating by parts correctly.

1 For finding recognising $\int_0^{\frac{1}{2}} \frac{x^{n+1}}{(1-x)^2} dx \approx 0$

For large values of n $\int_0^{\frac{1}{2}} \frac{x^{n+1}}{(1-x)^2} dx \approx 0$ from part (iv) and so

$$I_n = n\int_0^{0\cdot5} \frac{1}{(1-x)^2}\, dx - (n+1)\int_0^{0\cdot5} \frac{x}{(1-x)}\, dx$$

$$= n\int (1-x)^{-2}\, dx - (n+1)\int x(1-x)^{-2}\, dx$$

$$= \left[n\frac{(1-x)^{-1}}{-1\times -1} \right]_0^{0\cdot5} \;-\; (n+1)\left[\frac{(1-x)^{-1}}{-1\times -1}\times x - \int \frac{(1-x)^{-1}}{-1\times -1}\times 1\, dx \right]$$

$$= \left[\frac{n}{1-x} \right]_0^{0\cdot5} \;-\; (n+1)\left[\left[\frac{x}{1-x} \right]_0^{0\cdot5} - \int_0^{0\cdot5} \frac{1}{(1-x)}\, dx \right]$$

$$= \left[\left(\frac{n}{1-\frac{1}{2}} \right) - \left(\frac{n}{1} \right) \right] - (n+1)\left[\left(\frac{\frac{1}{2}}{1-\frac{1}{2}} \right) - \left(\frac{0}{1} \right) + \left[\ln(1-x) \right]_0^{0\cdot5} \right]$$

$$= [n] - (n+1)\left[1 + \left(\ln\frac{1}{2} - \ln 1 \right) \right]$$

$$= n - n - 1 - (n+1)\ln\frac{1}{2}$$

$$= (n+1)\ln 2 - 1$$

Marker's Comments

Despite having previous parts to assist, few students could resolve the integral into 3 components and use the previous results to indicate

$$\int_0^{\frac{1}{2}} \frac{x^{n+1}}{(1-x)^2}\, dx \approx 0$$

for large values of n.

Many attempted to substitute $n=\infty$ thus entirely missing the point of the question.